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Relatively Prime Domination in Discrete and Ascending Topological Graphs

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ABSTRACT

Let (X, τ) be a topological space. The graph $G_\tau = (V, E)$ is called a topological graph if $V = \{u: u \in \tau, u \neq \emptyset, X\}$ and $E = \{uv \in E\ (G_\tau)$ if $u \cap v \neq \emptyset, u \neq v$ and $u, v \in \tau\}$. In this paper, we explore the relatively prime domination number in discrete topological graphs and ascending topological graphs. Additionally, we investigate the relatively prime domination number in the context of switching between discrete and ascending topological graphs, as well as in modified graphs resulting from the deletion or addition of vertices in both discrete and ascending topological graphs.

Keywords: *Domination, Reconstruction.*

1. Introduction

Atopology [1] on a non-empty set X is a collection τ of subsets of X having the following properties:$\emptyset$ and X are in τ; the union of the elements of any sub collection of τ is in τ; the intersection of the elements of any finite sub-collection of τ is in τ and A set X for which a topology τ has been specified is called a topological space. That is, a topological space is an ordered pair (X,τ)consisting of a set X and a topology τ on X. If τ consists the collection of all subsets of X is called the discrete topology and the collection consisting of X and $\emptyset$ is called the indiscrete topology or the trivial topology. A graph G consists of a non-empty finite set V(G) of elements called vertices and a finite set E(G) of distinct unordered pairs of distinct elements of V(G) called edges. The graph $G_\tau = (V, E)$ is called a topological graph if V = {u: u $\in \tau, u \neq \emptyset, X\}$ and E = {uv $\in$ E (G_τ) if u $\cap\, v \neq \emptyset, u \neq v$ and $u, v \in \tau\}$.

Researchers Ali Ameer Jabor and Ahmed Abd-Ali Omran explored the properties of discrete topological spaces, establishing that the domination number for a discrete topological graph is 2. They also delved into affection domination by manipulating edges or vertices in a Discrete Topological Graph. In our study, we have presented findings on the relatively prime domination number for discrete topological graphs and ascending topological graphs, demonstrating that it is zero. Additionally, we have discussed the concept of relatively prime domination for switching graphs and modified graphs obtained by deleting or adding edges or vertices.

2. Basic Definitions

Definition 1 [3] Deleting the Vertex: If v is a vertex of a graph G, we denote by $G - v$ the graph obtained from G by deleting the vertex v together with the edges incident on v.

Definition 2 [3] Deleting the Edge: If e is an edge of a graph G, we denote $G - e$ the graph obtained from G by deleting the edge e.

Definition 3 [4] Dominating Set: A subset S of V is said to be dominating set in G if every vertex in $V - S$is adjacent to at least one vertex in S.

Definition 4 [4] Domination Number: The domination number γ(G) is the minimum cardinality taken over all dominating set of G.

Definition 5 [5] Relatively Prime Dominating Set: Let G be a non-trivial graph. A set $S \subseteq V$ is said to be relatively prime dominating set if it is a dominating set with at least two elements and for every pair of vertices u and v in S such that $(\deg(u), \deg(v)) = 1$.

Definition 6 [5] Relatively prime Domination Number: The minimum cardinality taken over all relatively prime dominating set is called relatively prime domination number and it is denoted by $\gamma_{rpd}(G)$.If there is no such pair exist, then$\gamma_{rpd}(G) = 0$.

Definition 7 [5] Switching in Graphs: For a finite undirected graph G(V, E) and a subset σ ⊆ V, the switching of G by σ is defined as the graph $G^{\sigma}(V, E')$ which is obtained from G by removing all edges between σ and V-σ. For $\sigma = \{v\}$, we write G^{v}instead of $G^{\{v\}}$ and the corresponding switching is called as vertex switching

3. Degree of each vertex in the Discrete Topological Graph

Let (X, , τ) be the discrete topological space with $|X| \geq 3$ and G_{τ} be the discrete topological graph. In order to find the relatively prime domination number, we need to find the degree of each element.

Consider the discrete topological space with five elements.

Let $X = \{a, b, c, d, e\}$ and so $|\tau_X| = 2^5 = 32$.

Now by the definition of topological graph $|V(G_{\tau})| = 30$.

First we find the degree of the element of order 4.

For that, first we find the number of elements which are non-adjacent to it.

Let us represent the five elements in the set X as 1+1+1+1+1.

Here, the first four one represents the chosen element of order 4.

Remaining only one 1's is left.

This shows that, there is a possibleelementsof order 4, 3, 2 such that their intersection with the chosen element is nonempty.

Thus, the only possible element is of order 1.

Thus, number of elements of order 1 which are non-adjacent to the chosen element = $1\,C_1$=1

Therefore, number of elements non-adjacent to chosen element = 1

Hence degree ofelement of order 4 = $2^n - 2 - 1 - 1 = 2^n - 2^{-1} - 2$

Next, we find out the degree of an element of order 3.

Let the element be U. As before, 1+1+1+1+1 represent the five elements in the set X and the first three elements in U.

Remaining two 1's is left.

This shows that there are no possible elements of order 4, 3 such that intersection with U is non-empty.

Therefore, the only possible elements are of order 1 and 2

Now, number of elements of order 2 which are non-adjacent to U = $2C_2$ = 1

Number of elements of order 1 which are non-adjacent to U = $2\,C_1$ = 2

Therefore, number of elements non-adjacent to U = 1+ 2 = 3

Hence degree of $U = 2^5 - 2 - 3 - 1 = 2\ 5 - 2 - 4 = 2^5 - 2 - 2^2 = 2^5 - 2^2 - 2$

Next, we find the degree of an elements of order 2.

Let the element be U.

The first two 1's represent the elements in the set U.

Remaining three 1's is left.

Thus, there is no possible element of order 3 such that their intersection with U is non-empty.

Therefore, the only possible elements of order which are no- adjacent to U is 1, 2, 3

Number of elements of order 3 which are non-adjacent to U = $3C_3$ = 1

Number of elements of order 2 which are non-adjacent to U = $3C_2$ = 3

Number of elements of order 1 which are non-adjacent to U = $3C_1$ = 3

Therefore, number of elements which are non-adjacent to U = 1 + 3 + 3 = 7

Degree of $U = 2^5 - 2 - 8 = 2^5 - 2^3 - 2$

Finally, we find out the Degree ofelements of order 1.

Let the elements be U.

Remaining four 1's is left.

Hence the possible element of order which are non-adjacent to U is 4,3,2,1.

Number of elements of order 4 which are non-adjacent to U = $4C_4$ = 1

Number of elements of order 3 which are non-adjacent to U = $4C_3$ = 4

Number of elements of order 2 which are non-adjacent to U = $4C_2$ = 6

Number of elements of order21 which are non-adjacent to U = $4C_1$ = 4

Number of elements which are non-adjacent to U = 1+4+6+4 = 15

Therefore, degree of $U = 2^5 - 2 - 16 = 2^5 - 2^4 - 2$

Hence degree of an element of order 4 = $2^5 - 2^{5-4} - 2 = 2^5 - 2^1 - 2$

Degree of an element of order 3 = $2^5 - 2^{5-3} - 2 = 2^5 - 2^2 - 2$

Degree of an element of order 2 = $2^5 - 2^{5-2} - 2 = 2^5 - 2^3 - 2$

Degree of an element of order 1 = $2^5 - 2^{5-1} - 2 = 2^5 - 2^4 - 2$

In general, let X be a topological space of order $n, n \geq 3$ and $G_\tau$be the discrete topological graph.

Then, the degree of an element of order 1 = $2^n - 2^{n-1}$ -2

Degree of an element of order 2 = $2^n - 2^{n-2}$- 2

Degree of an element of order 3 = $2^n - 2^{n-3} - 2$

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Degree of an element of order n-2 = $2^n - 2^2 - 2$

Degree of an element of order n-1 = $2^n - 2 - 2$

Note: Here the degree of each element in the discrete topological graph is even.

4. Relatively Prime Domination Number

Throughout this section, n denotes the number of elements in $X; p$ denotes the number of vertices in the graph G_τ; r denotes the order of an open set.

Theorem 4.1. Let (X, τ) be the discrete topological space with $|X| \geq 3$. Then the relatively prime domination number of discrete topological graph is $\gamma_{rpd}(G_\tau) = 0$.

Proof: Let (X, τ)be the discrete topological space with $|X| \geq 3$ and G_τ be the discrete topological graph**.** By above note, degree of each vertex inG_τ is even. Therefore, relatively prime dominating set does not exist for discrete topological graph and hence $\gamma_{rpd}(G_\tau) = 0$.

Remark: As relatively prime dominating set does not exist for discrete topological graph; we find the relatively prime domination number for switching graph of Discrete

Topological Graph.

Theorem 4.2. Let (X, τ) be the discrete topological space of order $n, n \geq 3$. hen

$$\gamma_{rpd}(G_\tau^U) = \begin{cases} 2 \; if \; (\deg U, \deg V) = 1, V \in V(G_\tau^U), U \subseteq V, \deg(U) + \deg(V) \geq p - 1 \\ 0 \quad\quad Otherwise \end{cases}$$

Proof: Let (X, τ)be the discrete topological space and G_τ be the discrete topological graph**.** We proceed by two cases.

Case 1:U be the vertex of order $r, 1 \leq r \leq n-2, n = |X|$.

In the resulting graph G_τ^U, degree of U is odd.Now, we choose an element in $V(G_\tau^U)$, say Vsuch that V contains the vertex U and $\deg(U) + \deg(V) \geq p - 1$.If $(\deg U, \deg V) = 1$, then relatively prime dominating set $= \{U, V\}$ and the relatively prime domination number is 2. If no such vertex exist in $V(G_\tau^U)$,then the relatively prime dominatingset does not exist,as we cannot choose an element V in $V(G_\tau^U)$ with order V less than order of U, because $\deg(U) + \deg(V) < p - 1$. Therefore,$\gamma_{rpd}(G_\tau^U) = 0$ in this case.

Case 2:U be the vertex of order $n - 2$.

In the resulting graph G_τ^U, degree of U is one. Now choose an element, say V in $V(G_\tau^U)$, of order $n - 2$. (If we choose an element of order less than n-2 in $V(G_\tau^U)$, then $\deg(U) + \deg(V) < p - 1$). Since $\deg(U) = 1$, let the element be $\{a\}$. Since $\deg(V) = p - 2$, and it contains the element $\{a\}$, we still left with an element to cover, say $\{b\}$. In order to cover $\{b\}$, we must choose an element contains $\{b\}$. As degree of all other elements except U has degree even, the relatively prime dominating set does not exist. Therefore,$\gamma_{rpd}(G_\tau^U) = 0$ in this case.

Thus,

$$\gamma_{rpd}(G_\tau^U) = \begin{cases} 2 \; if \; (\deg U, \deg V) = 1, V \in V(G_\tau^U) \; and \; U \subseteq V, \deg(U) + \deg(V) \geq p - 1 \\ 0 \quad\quad Otherwise \end{cases}$$

Theorem 4.3. Let (X, τ) be the topological space with $|X| \geq 3$. Then

$$\gamma_{rpd}(G_\tau - e) = \begin{cases} 2 \quad if \; (\deg U, 2^n - 4) = 1, \; where \; e \; is \; incident \; with \; the \; vertex \; U \\ 0 \quad\quad Otherwise \end{cases}.$$

Proof: Let (X, τ) be the topological space with $n \geq 3$ where $n = |X|$ and G_τ be the discrete topological graph. Let e be the edge removed from G_τ**.** Then degree of two elements say U and V which incident with the edge e is $\deg(U) - 1, \deg(V) - 1$. Therefore, $\deg(U) - 1 = \deg(V) - 1 =$ odd**.** Since the vertices of order $n - 1$covers $p - 1$ vertices, we take such a vertex say W**.**

Since $\deg(V) = $ odd and if $(\deg(U), \deg(W)) = (\deg(U), 2^n - 4) = 1$,relatively prime dominating setis $\{U, W\}$ and hence $\gamma_{rpd}(G_\tau^U) = 2$. If $(\deg(U), \deg(W)) \neq 1$, then relatively prime dominating set does not exist and so $\gamma_{rpd}(G_\tau - e) = 0$.

Theorem 4.4. Let (X, τ)be the discrete topological space with $|X| \geq 3$. Then

$$\gamma_{rpd}(G_\tau - v) = \begin{cases} 2 \quad if\ (\deg U, 2^n - 4) = 1, \quad where\ e\ is\ incident\ with\ the\ vertex\ U \\ 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad Otherwise \end{cases}.$$

Proof: Let (X, τ)be the discrete topological space with $n \geq 3$ where $n = |X|$ and G_τ be the discrete topological graph. Consider the graph $G_\tau - v$. Here the vertices which are adjacent with the vertex v has odd degree. Denote the vertices which have odd degree by $U_i, i = 1, 2, \ldots, n - 2$. Note that U_i is a vertex of order i in G_τ. Now, choose a vertex, say W of degree $p - 1$. If $(\deg(U_i), \deg(W)) = 1$, for some U_i, then the relatively prime dominating set is $\{U, W\}$ and hence $\gamma_{rpd}(G_\tau^U) = 2$. If $(\deg(U), \deg(W)) \neq 1$, for all U_i, then relatively prime dominating set does not exist.

Theorem 4.5. Let (X, τ) be the topological space with $|X| \geq 3$. Then

$$\gamma_{rpd}(G + e) = \begin{cases} 2\ if\ (\deg U, 2^n - 4) = 1, \quad where\ e\ is\ incident\ with\ the\ vertex\ U \\ 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad Otherwise \end{cases}$$

Proof: Let (X, τ)be the topological space with $n \geq 3$ where $n = |X|$ and G_τ be the discrete topological graph. Let e be the new edge added in G_τ.Then degree of two elements say U and V which are incident with the edge e is $\deg(U) + 1, \deg(V) + 1$, which is odd.Now if $(p - 1,\ \deg(U) + 1) = 1$, then the relatively prime dominating set exist and$\gamma_{rpd}(G_\tau + e) = 2$.Otherwise, the relatively prime dominating set does not exist.

5. Relatively Prime Domination on Ascending Topological Grah

In this section, we have discussed about the properties and domination, relatively prime domination number of Ascending Topological Graph. A topological space X is said to have an ascending chain if every open sets of X forms an ascending chain. A topological space which have an ascending chain is called as Ascending Topological Space.

Let (X,τ) be a topological space with $|X| \geq 3$ having ascending chain. Let G_τ be the corresponding topological graph. Since the open sets in τ are in ascending chain, the intersection between any two non-empty open sets in non-empty. Therefore, in the corresponding topological graph, adjacency exist between any two vertices and hence we get a complete graph.

Proposition 5.1. The graph G_τ is a null graph if $|X| \leq 2$.

Proof: Let (X,τ) be an ascending topological space and G_τ be the corresponding topological graph. If $|X| = 1$, then $\tau_x = \{\emptyset, X\}$. As we exclude the $\emptyset$, X in the construction of topological graph, we have $V(G_\tau) = \emptyset$.If $|X| = 2$, then $\tau_x = \{\emptyset, \{a\}, X\}$.Here $V(G_\tau) = \{a\}$.If $|X| \geq 3$, then $V(G_\tau)$contains at least two distinct vertices and adjacency exist, as they are in ascending chain in τ. Therefore, the graph G_τ is a null graph if $|X| \leq 2$.

Properties of Ascending Topological Graph

Let (X,τ) be ascending topological space with $|X| \geq 3$ and G_τ be the corresponding topological graph. Since G_τ is a complete graph,G_τ has the following properties

- Number of edges in G_τ is $\frac{p(p-1)}{2}$ where $p = |V(G_\tau)|$.
- G_τ is connected.
- G_τ has no cut vertex.
- G_τ is Hamiltonian.
- $G_\tau$is Eulerion if $|X| =$ even
- G_τ is a Complete Bipartite Graph
- Domination number of G_τ is 1

Theorem 5.2. Let (X,τ) be an ascending topological space with $|X| \geq 3$. Then$\gamma(G_\tau - v) = \gamma(G_\tau) = 1$ and $\gamma(G_\tau - e) = \gamma(G_\tau) = 1$.

Proof: Let (X,τ) be an ascending topological space with $|X| \geq 3$ and G_τ be the corresponding topological graph. Consider the graph $G_\tau - v$**.** Since G_τ is a complete graph, we again get the complete graph by removing a vertex. As$\gamma(G_\tau) = 1$, we hve $\gamma(G_\tau - v) = \gamma(G_\tau)$. Next, we consider the graph $G_\tau - e$. Let the vertices incident with $e$be v_i,v_j.Then $d(v_i,) = d(v_j) = p - 2$.But the remaining all vertices have degree $p - 1$.Therefore, a vertex with degree $p - 1$ covers all the vertices of G_τ. Hence $\gamma(G_\tau - e) = \gamma(G_\tau) = 1$.

Theorem 5.3. [5]Let (X,τ) be an ascending topological space with $|X| \geq 3$.Then $\gamma_{rpd}(G_\tau) = 0$ and $\gamma_{rpd}(G_\tau^v) = 0$.

Theorem 5.4. Let (X,τ) be an ascending topological space with $|X| \geq 3$.Then $\gamma_{rpd}(G_\tau - v) = 0$ and $\gamma_{rpd}(G_\tau - e) = 2$.

Proof: Clearly$\gamma_{rpd}(G_\tau - v) = 0$ as $G_\tau - v$ is a complete graph. Next, consider the graph $G_\tau - e$. Here degree of each vertex is either $p - 1$ or $p - 2$.The vertex of degree$p - 1$ covers all the vertices of $G_\tau - e$.In order to get a relatively prime dominating set, we choose a vertex of degree $p - 2$.Let us say the vertices of degree $p - 1$ and $p - 2$ be u and v respectively. Therefore, relatively prime dominating set = $\{u, v\}$and thus relatively prime domination number is 2.

Theorem 5.5. Let (X,τ) be an ascending topological space with $|X| \geq 3$.Then$\gamma_{rpd}((G_\tau - v)^v) = 0$.

Proof: Let (X,τ) be an ascending topological space with $|X| \geq 3$andG_τ be the corresponding topological graph. Consider $G_\tau - v$. Then degree of each vertices is equal, say $p - 2$. Let v be the switching vertex. Then in the graph $(G_\tau - v)^v$, the switching vertex has degree zero and the remaining vertices have degree p−3. Therefore, relatively prime dominating set does not exist. Therefore $\gamma_{rpd}((G_\tau - v)^v) = 0$.

6. Conclusion

In this research article, we explored the relatively prime domination number concerning discrete topological graphs and ascending topological graphs. Furthermore, we delved into the notion of relatively prime domination within switching graphs and modified graphs resulting from edge or vertex deletions or additions. Analogously, we can investigate the graph-theoretical characteristics of topological spaces by transforming them into standard topological graphs.

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